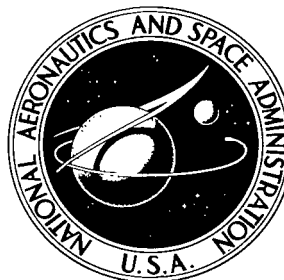


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# LYAPUNOV STABILITY CRITERIA FOR RANDOMLY SAMPLED SYSTEMS

*by Leonard Tobias*

*Electronics Research Center  
Cambridge, Mass.*

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# LYAPUNOV STABILITY CRITERIA FOR RANDOMLY SAMPLED SYSTEMS

By Leonard Tobias  
Electronics Research Center

## SUMMARY

This study is concerned with the asymptotic behavior of systems in which random sampling occurs; they are studied by a stochastic Lyapunov function method. The control loops under consideration consist of a random sampler (a sampling device which closes at a set of statistically described times in lieu of periodic intervals), a zero order hold, a linear plant, and a feedback element. Sampled systems are modelled randomly when sampler imperfections, such as jitter or skipping, occur or when a single computer or communications link is a component of multiple control loops (that is, when the availability times of the computer or communications link to a particular control loop are random). This type of model has also been suggested for a human operator performing a compensatory tracking function.

Improved stability criteria are given for systems whose inputs are identically zero for all time. When the feedback element is linear, sufficient conditions for asymptotic mean-square stability and asymptotic stability with probability one are obtained and compared. Necessary and sufficient conditions are also presented; these are used to analyze the value of the sufficient conditions. Intersample behavior is studied and results are presented for both stable and unstable plants. Numerical results illustrate the applicability and utility of the criteria presented and describe some interesting phenomena such as jitter-stabilized systems. When random inputs are present, a general method is given for the computation of the asymptotic mean-square output at sample instants. This method is illustrated by a computer program for a general second-order system.

A randomly sampled Lure problem is studied and sufficient conditions for asymptotic mean-square stability and asymptotic stability with probability one are derived.

## INTRODUCTION

This study is concerned with the asymptotic behavior of systems in which random sampling occurs. A random sampler is a sampling device which closes at a set of times  $t_1, t_2, \dots$ , which are known only in a statistical sense.

Random sampling may occur unintentionally due to physical imperfections in the sampling device. That is, a system is desired in which periodic sampling occurs, but device imperfections prevent this from happening. These imperfections are usually of two types - imperfections in timing (jitter) and imperfections in closing (skipping). In the case of jitter, the sampler does not close precisely at the periodic sample times, but in some neighborhood of these times. Skipping occurs when the sampler is supposed to close at times,  $nT$ , but fails to make connection at some of these times. If probability information is available regarding the jitter width or the probability of skipping, then precise stability conditions can be formulated. (Clearly these will be different from the conditions for stability of periodically sampled systems).

Intentionally randomly sampled systems may occur for many reasons. Recent years have seen the increased use of digital computers as components in control systems. This has introduced new problems, since the digital computer accepts and supplies data at discrete time instants; also, the same computer is often a component of many control loops. Because one system may require the full use of the computer during critical stages, the availability times of the computer to a particular process may be described only in some statistical sense.

Space systems seems to be an area of fruitful applications of random sampling analysis techniques. Consider an unmanned Earth-controlled vehicle. Suppose it is desired to maintain a constant velocity as the vehicle moves about the irregular, hilly surface of the Moon. The vehicle carries a velocity sensor; velocity information is sent via a communications link to an Earth-based human operator, who sends velocity commands back to the vehicle. However, via the same link, information regarding surface rocks and soil, photographs of the lunar surface, etc., must be transmitted. Due to the multiple systems it must handle and due to the unknown time of processing of information from each task, the availability times of the link to a particular task are random. Stability analysis is necessary to determine the feasibility of performing a particular task.

There is a considerable active engineering interest in obtaining mathematical models of the human operator in compensatory tracking functions. Some consideration has been given by G. Bekey et al. (ref. 1) to modelling the human operator with randomly sampled systems. Young et al. (ref. 2), have proposed a system for modelling of eye-tracking. Their model includes two kinds of control. The first, called pursuit control, is an open-loop control which is a continuous function of the rate of target movement. Saccadic (or jump) control is the second type of control which is introduced to account for large discrepan-

cies in eye and target position. It is closed loop and is modelled as a sampler and hold followed by non-linearities and delays. Two types of sampler models have been proposed – target-synchronized and non-synchronized. The former implies that there is a stochastic delay between the time the target moves and the sampling time. The latter implies that samples are taken whether or not retinal error has occurred.

Hence, an analysis of randomly sampled systems is required. Some effort has been made in this direction, but much more remains to be investigated. Brown (ref. 3) has investigated sampled systems with jitter; filtering and control aspects of randomly sampled systems are discussed in Chang (ref. 4).

The class of systems depicted in Figure 1 is considered. The zero-order hold is a device which accepts the sample  $e(t_i)$  as input at time  $t_i$ , and has an output  $e(t_i)$  for  $t$  in the interval  $[t_i, t_{i+1})$ . Plants under consideration will be linear. The first-order plant will be studied, followed by multiple-order plants (computer implementation for multiple-order stability conditions will be carried out). Feedback will, for the most part, be linear, but scalar non-linear elements are also considered. Asymptotic stability results will be presented for systems in which the input is identically zero; also, mean-square output behavior will be examined when the input is a stationary random process.

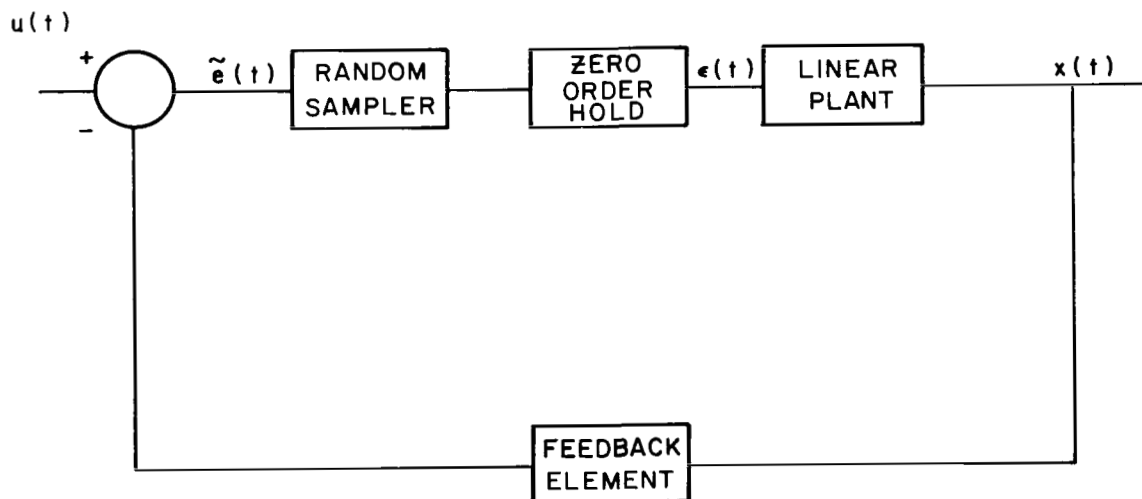


Figure 1.- The general randomly sampled linear system

The method of approach is a stochastic Lyapunov one; that is, as in the deterministic sampled problem, one seeks a non-negative functional  $V_n = V(x_n)$  of the solution paths  $\{x_n\}$  such that  $V_{n+1} - V_n$  is non-positive. The Lyapunov functions used ( $|x|^2$  for the scalar linear case,  $x'Wx$  for the multivariable linear case and  $x'Hx + q \int_0^\sigma f(\alpha) d\alpha$  for the Lure problem) have also been used for deterministic problems; however, the introduction of randomness changes the framework of stability statements to a probabilistic one; also, system performance may be radically altered by introduction of randomness.

In the following section, areas of application of this stability analysis are presented. Then results for undriven systems are presented followed by the driven results; the techniques are then used in non-linear systems. Some of the results for undriven systems can be found in reference 5; for proofs of the conditions stated in this report, see author's doctoral thesis (ref. 6).

The contributions of Professor Leonard Shaw of the Polytechnic Institute of Brooklyn and Professor Harold Kushner of Brown University in the development of the theory of stochastic control, in particular, as pertaining to the work presented in this document, are gratefully acknowledged by the author; the author is indebted to them for many helpful discussions and suggestions during the course of this work.

#### STABILITY OF LINEAR UNDRIVEN SYSTEMS

This section is concerned with the class of systems which can be modelled as in Figure 1. The simplest of cases is considered first; the case when the plant is first-order linear and the feedback is a scalar constant  $K_2$ . The plant is represented by  $K_1/s+r_1$ . This has been redrawn as Figure 2. Precise definitions of the types of stability considered appear in the appendix, as well as some fundamental theorems. The following notational conventions are adopted.

N1: Let  $0 < t_1 < t_2 < \dots$  be the set of times at which the sampler closes

N2:  $\Delta_i \triangleq t_{i+1} - t_i$

N3:  $f_{\Delta_i}(x) \triangleq$  probability density function of  $\Delta_i$

N4:  $\phi_i(s) \triangleq$  characteristic function of  $f_{\Delta_i}$

$$= E[e^{-s\Delta_i}] = \int_0^\infty f_{\Delta_i}(x) e^{-sx} dx$$

N5:  $K = K_1 K_2$

$$N6: X_n = X(t_n)$$

Assume that (A1)  $\Delta_i$  are independent and identically distributed and (A2)  $t_0 = 0$  wpl and  $x(0) \neq 0$  wpl.

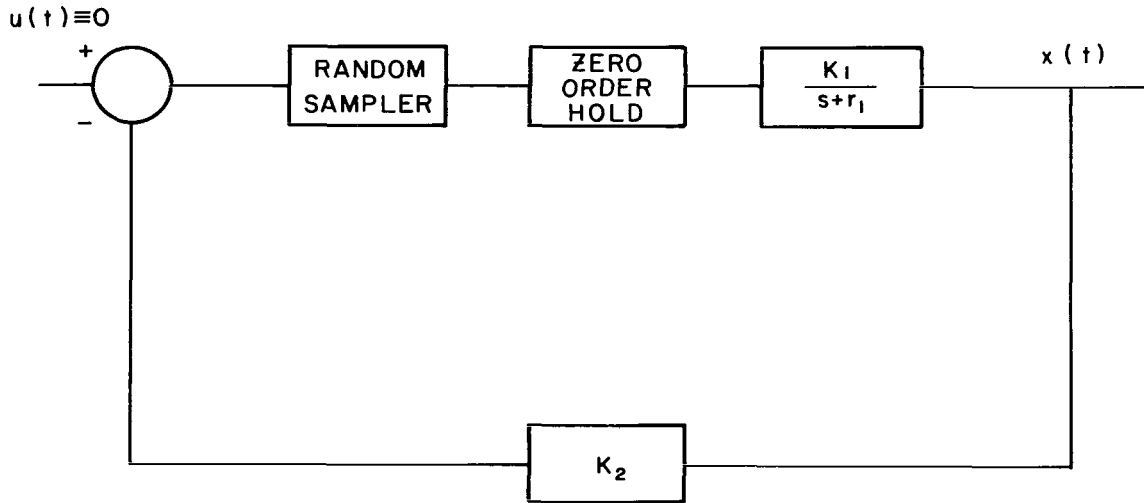


Figure 2.- Basic system for linear first-order undriven case

Theorem 1- If  $r > 0$ , a necessary and sufficient condition for the system in Figure 2 to be asymptotically stable of order  $s$  is

$$E |(1+K/r)e^{-r\Delta} - K/r|^s < 1. \quad (1)$$

If  $r < 0$ , we require the additional assumption that (A3)  $\Delta_i < T_m$ , where  $T_m < \infty$  is a constant independent of  $i$ . Note that for the case  $s = 2$  we have asymptotic mean-square stability and the condition (1) can be written

$$(1+K/r)^2 \phi(2r) - 2K/r(1+K/r)\phi(r) + (K/r)^2 < 1. \quad (2)$$

The inequality (2) can be found in Kalman (ref. 7) and Leneman (ref. 8). However, the cases when  $s \neq 2$  and  $r < 0$  are not treated there. The condition for stability of order,  $s$ , can, via the Doob Martingale Convergence Theorem, be used to establish a sufficient condition for asymptotic stability wpl.

Theorem 2- A sufficient condition for the system in Figure 2 to be asymptotically stable wpl is that for some  $s > 0$ , the system is asymptotically stable of order  $s$ .

Note that the condition for mean-square stability is not necessary; that is, it is possible for the system to be stable of order  $s < 2$  but not for  $s = 2$ .

To illustrate how one can use these conditions, consider Eq. (2). After some manipulation it becomes:

$$\left[ \left( \frac{K}{r} \right) - \frac{\phi(r) - \phi(2r)}{1 - 2\phi(r) + \phi(2r)} \right]^2 < \left[ \frac{1 - \phi(r)}{1 - 2\phi(r) + \phi(2r)} \right]^2$$

From the above, we deduce that for asymptotic mean-square stability K must lie in the range:

$$-1 < K < \frac{r(1 - \phi(2r))}{1 - 2\phi(r) + \phi(2r)} \quad (3)$$

We now consider how the range of K is affected by parameter variation of the densities.

Example 1, Exponential.— For exponential sampling with parameter  $\lambda$ ,

$$f_{\lambda}(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

It is readily shown that  $\phi(s) = \frac{\lambda}{\lambda + s}$

From Eq. (10),

$$\begin{aligned} K_{\max} &= \frac{r(1 - \phi(2r))}{1 - 2\phi(r) + \phi(2r)} \\ &= \frac{r(1 - \frac{\lambda}{\lambda + 2r})}{(1 - \frac{2\lambda}{\lambda + r} + \frac{\lambda}{\lambda + 2r})} \end{aligned}$$

and via some simple algebra

$$K_{\max} = \lambda + r$$

Thus,  $\partial K_{\max} / \partial \lambda = 1$ ; we can show that  $E(x) = 1/\lambda$ . Thus, we have demonstrated the reasonable result that if the mean sampling time decreases, the maximum allowable gain for stability increases (for the first-order system under discussion).

Example 2, Uniformly distributed jitter over a time range of  $\delta$  centered about a nominal sampling interval T.— From the appendix

$$\phi(s) = e^{-sT} \frac{\sinh(s\delta/2)}{s\delta/2}, \text{ and using Eq. (3)}$$



$$K_{\max} = \frac{r(1 - e^{-2rT} \frac{\sinh r \delta}{r\delta})}{1 - 2e^{-rT} \frac{\sinh r \delta/2}{(r\delta/2)} + e^{-2rT} \frac{\sinh r \delta}{r\delta}}$$

Note that when  $\delta = 0$

$$K_{\max} = \frac{r(1+e^{-rT})}{(1-e^{-rT})}$$

which is the standard result obtained by conventional periodic sampling analysis. A straightforward but tedious calculation shows that

$$\begin{aligned} \frac{\partial K_{\max}}{\partial \delta} = \frac{r^2}{m^2} & \left\{ \left[ -4 \sinh \left[ \frac{r\delta}{2} \right] - 2r\delta \cosh \left[ \frac{r\delta}{2} \right] \right] e^{-rT} \right. \\ & + [2 \sinh(r\delta) - 2r\delta \cosh(r\delta)] e^{-2rT} \\ & + \left[ 4 \sinh \left[ \frac{r\delta}{2} \right] \cosh(r\delta) + 2 \cosh \left[ \frac{r\delta}{2} \right] \sinh(r\delta) \right] e^{-3rT} \\ & \left. - [2 \sinh(r\delta) \cosh(r\delta)] e^{-4rT} \right\} \end{aligned}$$

$$\text{where } m = r\delta - 4e^{-rT} \sinh \left[ \frac{r\delta}{2} \right] + e^{-2rT} \sinh(r\delta)$$

The above indicates analytically the manner in which the maximum gain varies from a given  $T$  as a function of the jitter.

Before leaving the scalar case, let us briefly examine the case of non-linear feedback. Suppose that in lieu of the previously considered constant feedback gain  $K_2$ , we consider a non-linear feedback element  $K_2 f(\cdot)$ , where  $f$  satisfies

$$0 \leq \mu_n = \frac{f(x_n)}{x_n} \leq \mu < \infty \quad (4)$$

It is readily shown that if the system is asymptotically stable of order  $s$  when the feedback is the constant  $K_2\mu$ , then it is stable for all scalar non-linearities satisfying Eq. (4).

## MULTIPLE ORDER SYSTEMS

### Sufficient Conditions

We now consider the system of Figure 3. For  $t \in (t_n, t_{n+1})$ ,

$$\frac{dx}{dt} = Ax - m\epsilon_n, \quad (5)$$

where  $x$  and  $\epsilon_n$  are  $N$  vectors,  $m$  is a scalar, and  $A$  is an  $N \times N$  matrix.

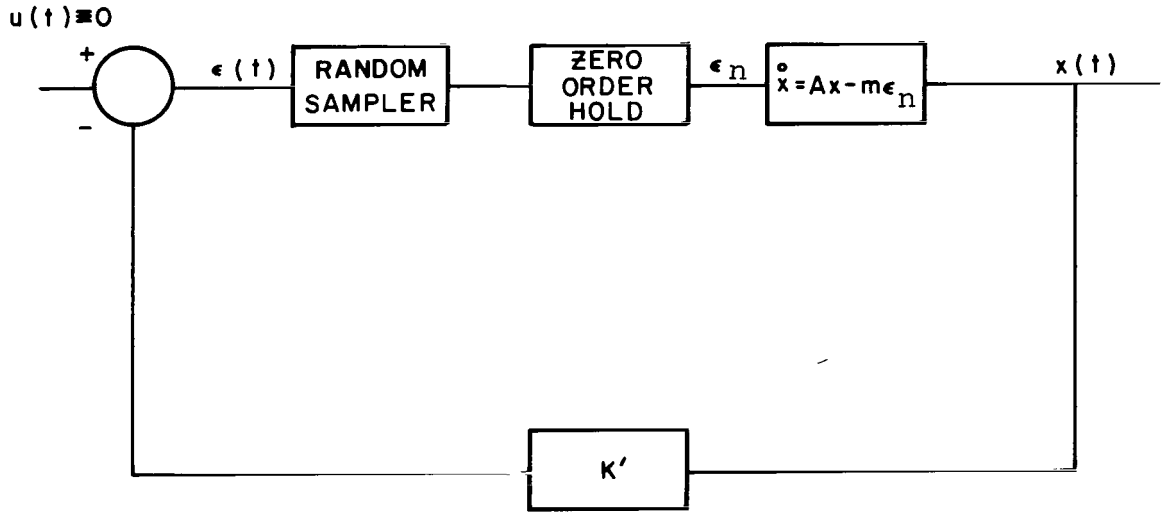


Figure 3.- Basic system for linear multiple-order case

Let  $K'$  be the feedback  $N$  vector and define  $C = mK'$ . Thus we may rewrite Eq. (5) as

$$\frac{dx}{dt} = Ax - C x_n \quad (6)$$

Hence 
$$x(t) = e^{A(t-t_n)} x_n + \left[ \int_{t_n}^t e^{A(t-\tau)} d\tau \right] C x_n$$

and at sample instants, 
$$x_{n+1} = (e^{A\Delta_n}(I - A^{-1}C) + A^{-1}C)x_n$$

or 
$$x_{n+1} = A_n x_n \quad (7)$$

$$\text{where } A_n = e^{A\Delta_n}(I-A^{-1}C) + A^{-1}C \quad (8)$$

We restrict attention in the multiple-order case to mean-square stability and stability with probability one. Recall that in the first-order case, a necessary and sufficient condition for mean-square stability is that  $E(\alpha_n^2) < 1$ , where  $\alpha_n = (1+K/r)e^{-r\Delta_n} - K/r$ . One might suspect that the extension to multidimensions is that the eigenvalues of the matrix  $E(A_n'A_n)$  be in the unit circle. The above is a sufficient condition; that it is not necessary will be clear from the ensuing theorems.

Theorem 3- Assume A1 and A2. In addition, assume A3 if any of the eigenvalues of the matrix  $E(A_n'A_n)$  lie on or outside the unit circle. A sufficient condition for the randomly sampled systems of Figure 3 to be asymptotically mean-square stable is that all the eigenvalues of the matrix  $E(A_n'A_n)$  lie in the unit circle.

$$\text{Now let } H = E[e^{A\Delta_i}] (I-A^{-1}C) + A^{-1}C \quad (9)$$

$$G_i = (e^{A\Delta_i} - E(e^{A\Delta_i})) (I-A^{-1}C) \quad (10)$$

Using this notation, we can write Eq. (7) as

$$X_{n+1} = (H+G_n)X_n$$

where  $E[G_n] = 0$ . Hence we have represented the system as a deterministic system  $H$  with random perturbation  $G_n$ . Suppose that the deterministic system

$$Y_{n+1} = HY_n \quad (11)$$

is asymptotically stable. Let  $W$  be the solution of  $H'WH - W = -Q$  for some positive definite  $Q$ . Then it can be shown that the perturbed system is asymptotically stable wpl if  $E[G_n'WG_n] - Q < 0$ . A computer study was undertaken to illustrate the application of this criterion. This study will be described presently but first necessary and sufficient conditions for asymptotic mean-square stability will be presented; these can be used to gauge the merit of the sufficient conditions.

One might naturally ask at this point, why bother with sufficient conditions at all if necessary and sufficient conditions are available? The answer is that the sufficient conditions are easier to apply; also Lyapunov functions may be known for the unperturbed system which may still be valid for

randomly perturbed systems. Finally, the methods of proof for the sufficient conditions can be extended to non-linear situations.

Let us then briefly consider necessary and sufficient conditions in light of the above and then turn our attention to the computer study. Necessary and sufficient conditions for asymptotic mean square stability of discrete systems were first obtained by Bharucha (ref. 9) in terms of Kraonecker products.

Consider the equation  $E(A'WA) - W = -Q$  where  $W$  and  $Q$  are positive definite and symmetric. Let the  $n(n+1)/2$  dimensional vector  $W$  and  $Q$  denote vectors composed of all the elements of  $W$  and  $Q$ , respectively. (Note that the vectors  $W$  and  $Q$  are not  $n^2$  vectors because  $W$  and  $Q$  are symmetric.) Then,  $E(A'WA) - W = -Q$  may be written as

$$AW - W = Q$$

Then a necessary and sufficient condition for asymptotic mean-square stability is that the roots of  $A$  lie in the unit circle.

#### Computer Results

The purpose of the computer study is to illustrate the use of the derived necessary and sufficient conditions and to determine the usefulness of sufficient ones. For the computer study, attention was focused on the system shown in Figure 4. The system is second order with two negative real roots  $r_1$  and  $r_2$ , and the feedback is a constant gain  $K$ . The system equation is

$$\frac{d^2x}{dt^2} - (r_1+r_2)\frac{dx}{dt} + r_1r_2 x = -K x_n \quad \text{for } t \in (t_n, t_{n+1}) \quad (12)$$

Let

$$y = \frac{dx}{dt} \quad (13)$$

$$\beta = K/r_1r_2 \quad (14)$$

$$\alpha = 1 + \beta \quad (15)$$

$$p = \frac{1}{r_2 - r_1} \quad (16)$$

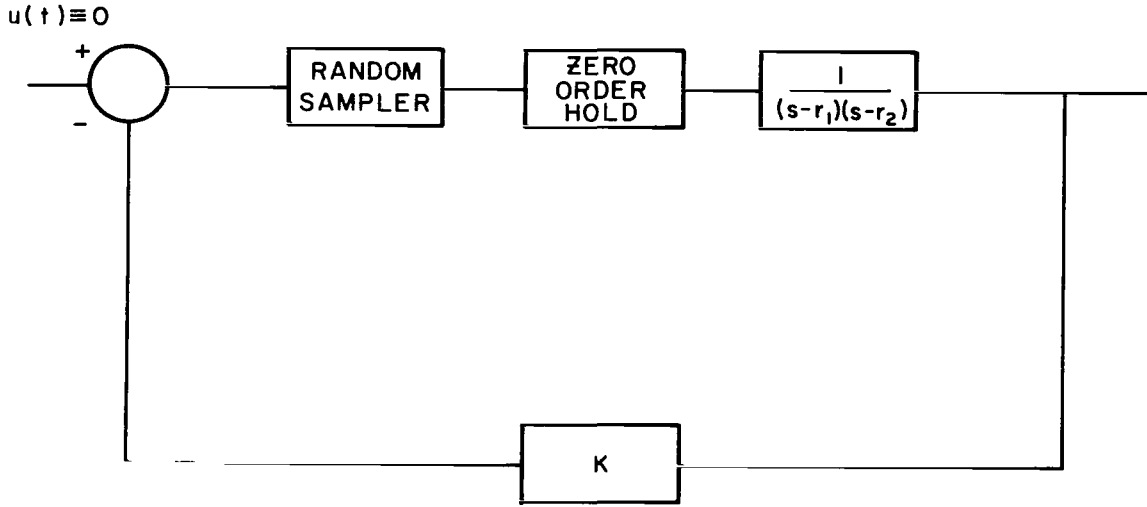


Figure 4.- The second-order plant with two negative real roots

At sample instants, it can be shown that the above system is given by

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = A_n \begin{pmatrix} x_n \\ y_n \end{pmatrix} \quad (17)$$

where

$$A_n = \begin{pmatrix} \alpha p^{r_2} e^{r_1 \Delta} - \alpha p r_1 e^{r_2 \Delta} - \beta & - p e^{r_1 \Delta} + p e^{r_2 \Delta} \\ \alpha p r_1 r_2 e^{r_1 \Delta} - \alpha p r_1 r_2 e^{r_2 \Delta} & - p r_1 e^{r_1 \Delta} + r_1 p e^{r_2 \Delta} \end{pmatrix} \quad (18)$$

If the following random coefficients are needed for the computation of  $E(A'WA)$  and  $A$ :

$$E(a_{11}^2) = (\alpha p)^2 \{ r_2^2 \phi(2r_1) - 2r_1 r_2 \phi(r_1 + r_2) + r_1^2 \phi(2r_2) \} \\ - 2\alpha \beta p \{ r_2 \phi(r_1) - r_1 \phi(r_2) \} + \beta^2$$

$$E(a_{11}a_{12}) = -\alpha p^2 \{r_2 \phi(2r_1) - (r_1+r_2)\phi(r_1+r_2) + r_1 \phi(2r_2)\} \\ - \beta p \{-\phi(r_1) + \phi(r_2)\}$$

$$E(a_{11}a_{21}) = (\alpha p)^2 r_1 r_2 \{r_2 \phi(2r_1) - (r_1+r_2)\phi(r_1+r_2) + r_1 \phi(2r_2)\} \\ - \beta \alpha p r_1 r_2 \{\phi(r_1) - \phi(r_2)\}$$

$$E(a_{11}a_{22}) = \alpha p^2 \{-r_1 r_2 \phi(2r_1) + (r_1^2+r_2^2)\phi(r_1+r_2) - r_1 r_2 \phi(2r_2)\} \\ - \beta p \{-r_1 \phi(r_1) + r_2 \phi(r_2)\}$$

$$E(a_{12}^2) = p^2 \{\phi(2r_1) - 2\phi(r_1+r_2) + \phi(2r_2)\}$$

$$E(a_{12}a_{21}) = -\alpha p^2 r_1 r_2 \{\phi(2r_1) - 2\phi(r_1+r_2) + \phi(2r_2)\}$$

$$E(a_{12}a_{22}) = p^2 \{r_1 \phi(2r_1) - (r_1+r_2)\phi(r_1+r_2) + r_2 \phi(2r_2)\}$$

$$E(a_{21}^2) = (\alpha p r_1 r_2)^2 \{\phi(2r_1) - 2\phi(r_1+r_2) + \phi(2r_2)\}$$

$$E(a_{21}a_{22}) = \alpha p^2 r_1 r_2 \{-r_1 \phi(2r_1) + (r_1+r_2)\phi(r_1+r_2) - r_2 \phi(2r_2)\}$$

$$E(a_{22}^2) = p^2 \{r_1^2 \phi(2r_1) - 2r_1 r_2 \phi(r_1+r_2) + r_2^2 \phi(2r_2)\}$$

Hence  $E(A'WA) - W = Q$  may be written as

$$\begin{pmatrix} E(a_{11}^2)w_{11} + 2E(a_{11}a_{21})w_{12} & E(a_{11}a_{12})w_{11} + E(a_{11}a_{22}) \\ & + a_{21}a_{12})w_{12} + E(a_{21}a_{22})w_{22} \\ E(a_{11}a_{12})w_{11} + E(a_{11}a_{22}) & E(a_{12}^2)w_{11} + 2E(a_{12}a_{22})w_{12} \\ & + a_{21}a_{12})w_{12} + E(a_{21}a_{22})w_{22} & + E(a_{22}^2)w_{22} \end{pmatrix} \\ - \begin{pmatrix} w_{11} & w_{12} \\ w_{12} & w_{22} \end{pmatrix} = - \begin{pmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{pmatrix}$$

In column vector form,  $AW - W = Q$

$$\begin{pmatrix} E(a_{11}^2) & 2E(a_{11}a_{21}) & E(a_{21}^2) \\ E(a_{11}a_{12}) & E(a_{11}a_{22}) + E(a_{12}a_{21}) & E(a_{21}a_{22}) \\ E(a_{12}^2) & 2E(a_{12}a_{22}) & E(a_{22}^2) \end{pmatrix} \begin{pmatrix} w_{11} \\ w_{12} \\ w_{22} \end{pmatrix} - \begin{pmatrix} w_{11} \\ w_{12} \\ w_{22} \end{pmatrix} = - \begin{pmatrix} q_{11} \\ q_{12} \\ q_{22} \end{pmatrix}$$

The sampling considered was uniformly distributed jitter with a range  $\delta$  centered about a nominal sampling interval  $T$ . It should be clear from the above that the parameters in this study are  $r_1, r_2, T, \delta, K$ ; for a given set of such parameters, one can compute  $E(A'WA) - W$  (for a given  $W$ ) and also  $A$  by using the formulas given above.

In one series of runs  $r_1, r_2, T$  and  $K$  were fixed;  $\delta$  was varied and for each  $\delta$ , the matrix  $A$  was computed. Let  $\rho$  = maximum magnitude of any eigenvalue of  $A$ . Hence, plots of  $\rho$  vs  $\delta$  could be obtained. A sample plot is shown in Figure 5. The line  $\rho = 1$  represents a stability boundary. By the theorem derived earlier  $\rho < 1$  is a necessary and sufficient condition for mean-square stability. The line  $\delta = 2T$  is a physical boundary since if  $\delta > 2T$ , it means that there is a positive probability that the  $(n+1)$ st sample occurs prior to the  $n$ th sample. For the typical plot shown  $\rho_1 = \rho(0)$  = the maximum eigenvalue without jitter.  $\delta_{\max}$  = the amount of jitter above which the system is unstable. Figure 6 represents an actual run for the set of parameters shown, where  $\rho_1 = .7$  and  $\delta_{\max} = .61$ . The interpretation of the graph is that the introduction of jitter causes an increase in  $\rho$ ; when a sufficient amount of jitter is introduced, the system is no longer mean square stable; hence, one may conclude that jitter is destabilizing for this set of parameters. Figure 7 is another plot of  $\rho$  vs.  $\delta$ . Here  $\rho_1 = 1.06$ , but as  $\delta$  increases,  $\rho$  decreases. In fact, in the region  $\delta \in [1.85, 3.152]$  the system is operating in the stable region. Hence, the conclusion is that jitter has stabilized a deterministically unstable system.

In order to check this surprising result, a digital simulation was made. A standard IBM 9094 random number routine, RANDU, generated a series of pseudo-random numbers uniformly distributed between zero and one. These numbers were then shifted and stretched so that a sequence of numbers with a uniform distribution  $\delta$  about a nominal sampling time  $T$  was obtained. Then the recursive equations (17) were solved iteratively. This was done for the initial conditions  $x_0 = 1$  and  $y_0 = 0$  and the same parameters as in Figure 7. The values  $x_n, y_n, \Delta_n$  were printed out for  $n = 1, \dots, 1000$ . With  $\delta = 0$  (no jitter),  $x_{1000} = 1.81 \times 10^{12}$  and

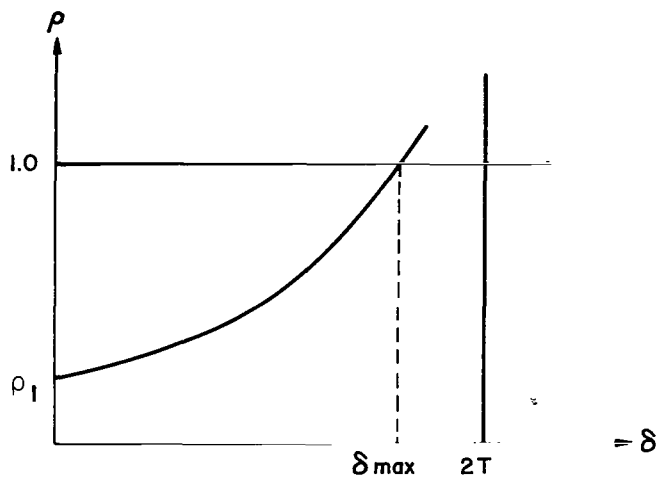


Figure 5.- A typical  
 $\rho$  vs  $\delta$  plot

Figure 6.- Actual  $\rho$  vs  $\delta$   
plot for  
 $K = 9.1$ ,  
 $r = -1.0$ ,  
 $r_2 = -2.0$ ,  
 $T = 1.0$

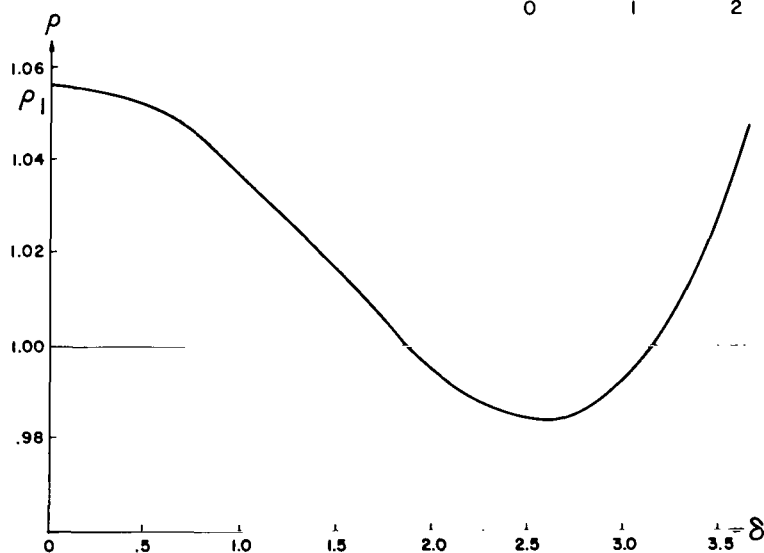
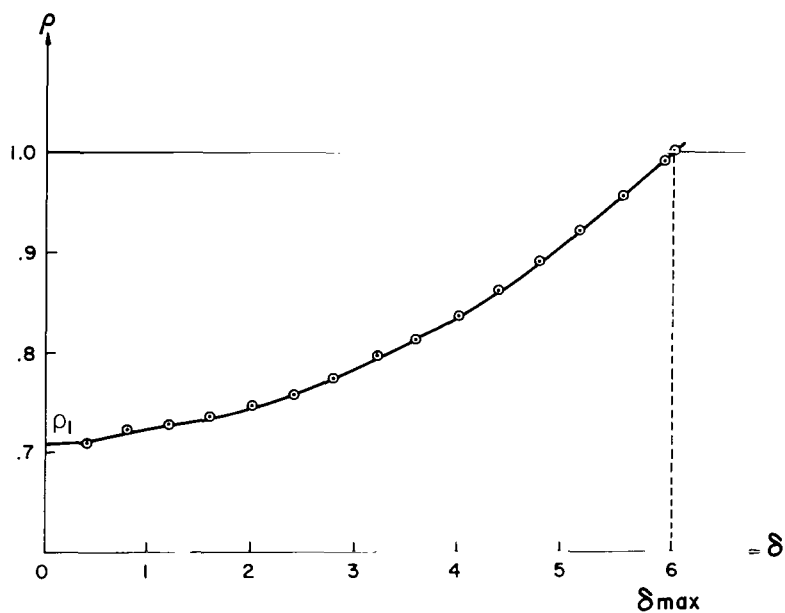


Figure 7.- Actual  $\rho$  vs  $\delta$   
plot for  
 $K = 6.0$ ,  
 $r_1 = -1.0$ ,  
 $r_2 = -4.0$ ,  
 $T = 2.0$



$y_{1000} = -8.32 \times 10^{11}$ . (Standard deterministic analysis shows that for  $K > 5.86$ , the system is unstable.) With  $\delta = 1.0$ ,  $x_{1000} = -.183$ ,  $y_{1000} = -.0562$ . Clearly the system is on the verge of instability. With  $\delta$  further increased to 2.5,  $x_n = y_n = 0.0 \times 10^{-38}$  for  $n > 400$ , indicating that the system is now stable.

Runs were also made for fixed  $r_1, r_2, T$  in which  $\delta_{\max}$  was plotted against  $K$ . Some results are in Figures 8 and 9.

A series of runs were made to determine the effect of the choice of Lyapunov function on the stability estimate. Recall that it has been shown that for a given  $W$  which is positive definite and symmetric, if

$$E(A'WA) - W < 0,$$

then the system is mean-square stable and asymptotically stable with probability one. Since the condition  $\rho < 1$  is necessary and sufficient, the goodness of the estimate using a particular  $W$  can be determined by finding the min  $\delta = \delta_W$  such that  $E(A'WA) - W < 0$  is no longer valid and comparing it with  $\delta_{\max}$ .

The following series of runs were made: fix  $r_1, r_2, K$  and  $T$  ( $K$  and  $T$  were picked so that with  $\delta = 0$  the system is stable). For  $Q = \begin{pmatrix} 1 & 0 \\ 0 & q_3 \end{pmatrix}$ , find  $W$  which satisfies  $A_T'WA_T - W = -Q$  where  $A_T$  is the matrix  $A$  when  $\delta = 0$ . Jitter is introduced incrementally and  $E(A'WA) - W$  is computed until  $\delta_W$  is determined. The run of Table I is for  $T = 1., K = 10., r = -1., r_2 = -2.$

TABLE I - RUN 1	
$q_3$	$\delta_W$
0.00	.40
.05	.35
.10	.31
.20	.26
.50	.18
1.00	.13

TABLE II - RUN 2	
$q_1$	$\delta_W$
.1	.04
.5	.10
1.0	.13
2.0	.26
10.0	.31
15.0	.34
20.0	.36

TABLE III - RUN 3	
$q_2$	$\delta_W$
.0	.05
.5	.11
.0	.13
-.5	.12
-.9	.07

From Figure 8,  $\delta_{\max} = .42$ . Hence, satisfactory bounds are obtained as  $q_3 \rightarrow 0$ . Table II illustrates a series of runs for  $Q = \begin{pmatrix} q_1 & 0 \\ 0 & 1 \end{pmatrix}$  and the same parameters  $r_1, r_2, K, T$  as Table I. Clearly, satisfactory bounds are obtained as  $q_1$  gets large. Hence, it seems that limiting values of  $q_1$  and  $q_3$  yield Lyapunov functions with the best estimate. Table III is included here to show that off-diagonal elements do not seem to improve the estimate. Note that

$$Q = \begin{pmatrix} 1 & q_2 \\ q_2 & 1 \end{pmatrix}.$$

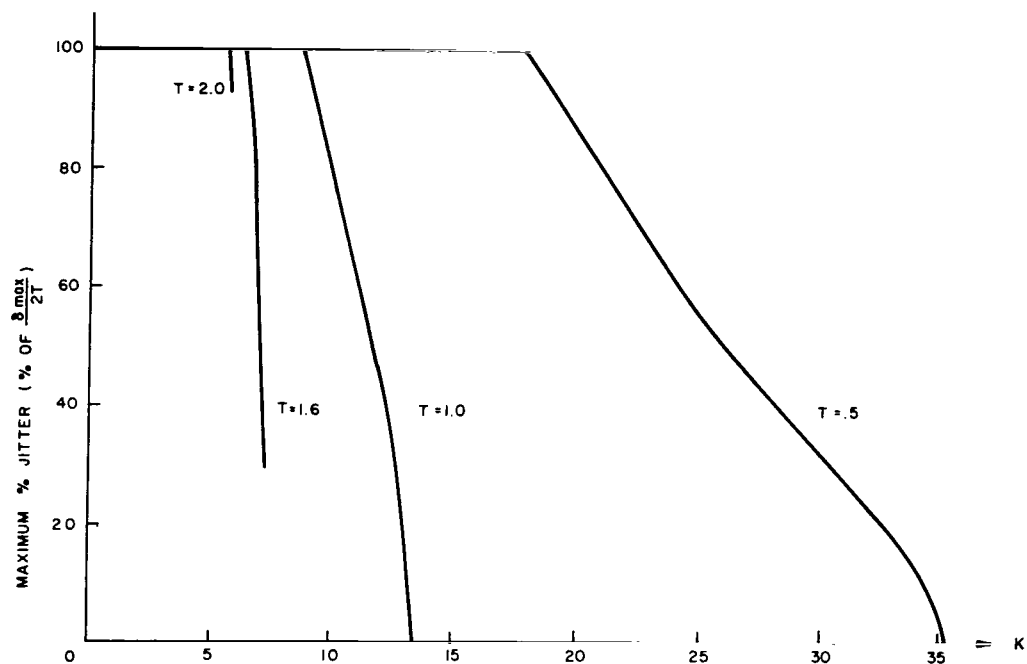


Figure 8.- Maximum % jitter vs gain K for  $r_1 = -10$ ,  $r_2 = -4.0$

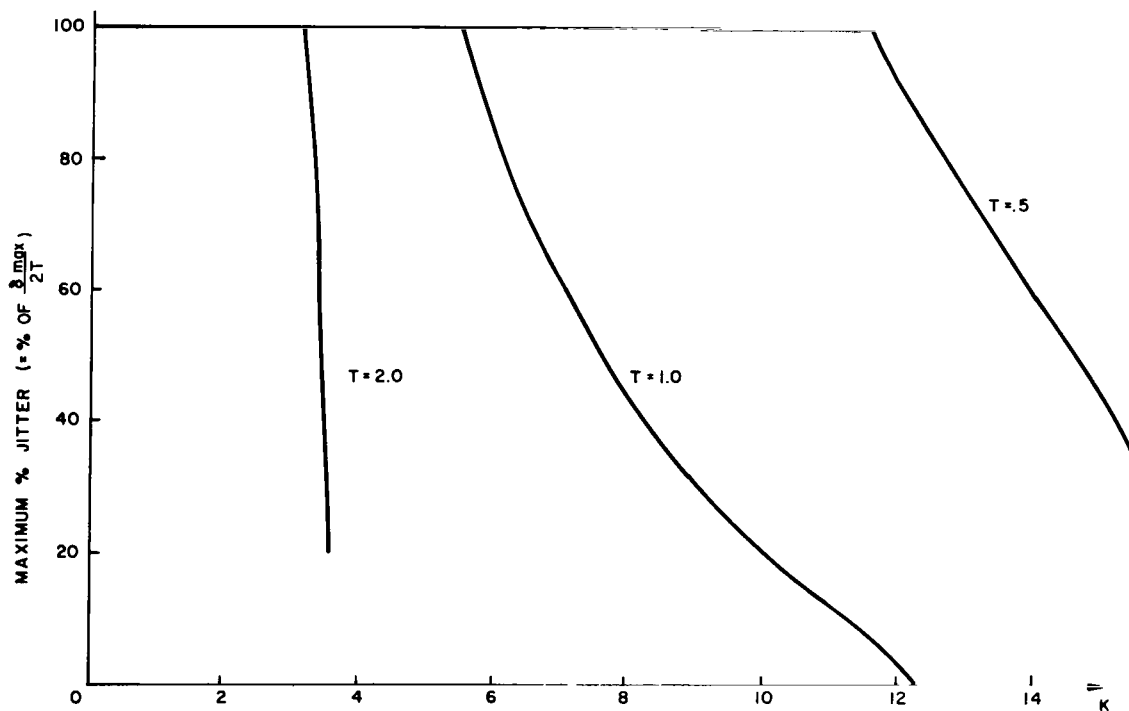


Figure 9.- Maximum % jitter for  $r_1 = -1.0$ ,  $r_2 = -2.0$

## LINEAR DRIVEN SYSTEMS

Suppose that the system previously considered is driven by a non-zero input  $u(t)$ . The class of inputs  $u(t)$  which will be considered are those which can be represented as outputs of a linear filter driven by white noise. The system under consideration is depicted in Figure 10 and is given in equation form below:

$$\frac{dx}{dt} = Ax + Bu - Cx_n \quad (19)$$

$$du = Du \, dt + Fdw \quad (20)$$

where  $\frac{dw}{dt}$  is white noise. (For a discussion of equations like Eq. (20), see reference 10.)

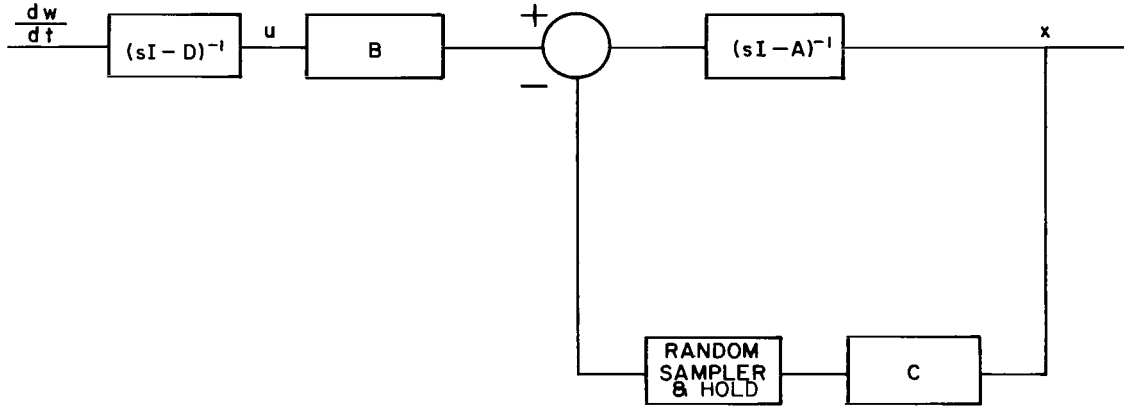


Figure 10.- System with gaussian input

At sample instants, Eqs. (19) and (20) become

$$\begin{aligned}
 x_{n+1} = & \left[ e^{A\Delta_n} + A^{-1}(I - e^{A\Delta_n})C \right] x_n + \left[ \int_{t_n}^{t_{n+1}} e^{A(t_{n+1}-\tau)} B e^{D(\tau-t_n)} d\tau \right] u_n \\
 & + \int_{t_n}^{t_{n+1}} e^{A(t_{n+1}-\tau)} B \left[ \int_{t_n}^{\tau} e^{D(\tau-\hat{\tau})} F dw(\hat{\tau}) \right] d\tau
 \end{aligned}$$

$$u_{n+1} = e^{D\Delta_n} u_n + \int_{t_n}^{t_{n+1}} e^{D(t_{n+1}-\tau)} E dw(\tau)$$

$$\text{Let } y = \begin{pmatrix} x \\ u \end{pmatrix}$$

$$A_n = \begin{pmatrix} e^{A\Delta_n} + A^{-1}(I-C)e^{A\Delta_n}C & \int_{t_n}^{t_{n+1}} e^{A(t_{n+1}-\tau)} B e^{D(\tau-t_n)} d\tau \\ 0 & e^{D\Delta_n} \end{pmatrix}$$

$$b_n = \begin{pmatrix} \int_{t_n}^{t_{n+1}} e^{A(t_{n+1}-\tau)} B \int_{t_n}^{\tau} e^{D(\tau-\hat{\tau})} F dw(\hat{\tau}) dt \\ \int_{t_n}^{t_{n+1}} e^{D(t_{n+1}-\tau)} F dw(\tau) \end{pmatrix}$$

Thus we can rewrite the above as

$$y_{n+1} = A_n y_n + b_n.$$

It can be shown that if  $y_{n+1} = A y_n$  is asymptotically mean-square stable, then

$$\lim E[y_n' \tilde{Q} y_n] = E[b' \tilde{W} b]$$

where  $\tilde{Q} > 0$  and  $\tilde{W} > 0$  satisfies

$$E(A' \tilde{W} A) - \tilde{W} = -\tilde{Q}.$$

Thus, we can compute the asymptotic mean-square behavior if we know certain undriven properties of the system (the Lyapunov function  $W$ ) and the parameters of the input. It is also possible to obtain estimates of intersample behavior but this will not be done here. The procedure is straightforward and will now be illustrated by a second-order example.

We consider the system shown below (Figure 11):

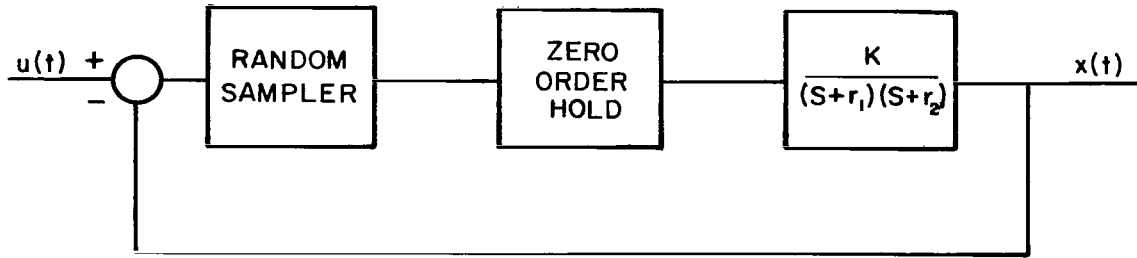


Figure 11.- A second-order driven system

The expression  $u(t)$  is a stationary process with autocorrelation

$$R_u(t) = e^{-r_3 t} \quad r_3 < 0 \quad (21)$$

We seek  $\lim_n E(x_n^2)$ .

Solution.- First, in order to apply the previous theorems and corollaries we need an appropriate model of the input process,  $u(t)$ . The model we seek is a differential system driven by white noise; it can be shown (Papoulis, ref. 11) that a suitable model is

$$du = r_3 u dt + \sqrt{-2r_3} dw$$

Thus, the input system at sample instants may be represented by

$$u_{n+1} = e^{r_3 \Delta} u_n + \sqrt{-2r_3} \int_{t_n}^{t_{n+1}} e^{r_3(t_{n+1}-\tau)} dw(\tau) \quad (22)$$

By slightly modifying the analysis in the previous computer study (p. 10) of a second-order example to include an input, we can obtain the following system of equations at sample instants:

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ u_{n+1} \end{pmatrix} = \begin{pmatrix} A_n & b_n \\ 0 & e^{r_3 \Delta_n} \end{pmatrix} \begin{pmatrix} x_n \\ y_n \\ u_n \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ h_n \end{pmatrix}$$

$A_n$  is defined by Eq. (18).

$$b_n = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} -\beta p(r_2 e^{r_1 \Delta} - r_1 e^{r_2 \Delta}) + \beta \\ -\beta p r_1 r_2 (e^{r_1 \Delta} - e^{r_2 \Delta}) \end{pmatrix}$$

$$h_n = \sqrt{-2r_3} \int_{t_n}^{t_{n+1}} e^{r_3(t_{n+1}-\tau)} dw(\tau)$$

$$\text{Let } A_n = \begin{pmatrix} A_n & b_n \\ 0 & e^{r_3 \Delta_n} \end{pmatrix}$$

$$C_n = e^{r_3 \Delta_n}$$

$h_n$  is a normal random variable with 0 mean and variance  $1-\phi(2r_3)$ . In addition, the following constants will be needed.

$$E(c^2) = \phi(2r_3)$$

$$E(b_1^2) = (\beta p)^2 \left[ r_2^2 \phi(2r_1) - 2r_1 r_2 \phi(r_1+r_2) + r_1^2 \phi(2r_2) \right] \\ - \beta^2 p(r_2 \phi(r_1) - r_1 \phi(r_2)) + \beta^2$$

$$E(b_2^2) = (\beta p)^2 (r_1 r_2)^2 \left[ \phi(2r_1) - 2\phi(r_1+r_2) + \phi(2r_2) \right]$$

$$E(b_1 b_2) = (\beta p)^2 r_1 r_2 \left[ r_2 \phi(2r_1) - (r_1+r_2) \phi(r_1+r_2) + r_1 \phi(2r_2) \right] \\ - \beta^2 p r_1 r_2 \{ \phi(r_1) - \phi(r_2) \}$$

$$E(a_{11}c) = \alpha p \{ r_2 \phi(r_1+r_3) - r_1 \phi(r_2+r_3) \} - \beta \phi(r_3)$$

$$E(a_{12}c) = -p \phi(r_1+r_3) + p \phi(r_2+r_3)$$

$$E(a_{21}c) = \alpha p r_1 r_2 \{\phi(r_1+r_3) - \phi(r_2+r_3)\}$$

$$E(a_{22}c) = -p\{r_1\phi(r_1+r_3) - r_2\phi(r_2+r_3)\}$$

$$E(b_1c) = -\beta p r_2 \phi(r_1+r_3) + \beta p r_1 \phi(r_2+r_3) + \beta \phi(r_3)$$

$$E(b_2c) = -\beta p r_1 r_2 \{\phi(r_1+r_3) - \phi(r_2+r_3)\}$$

$$E(a_{11}b_1) = -\alpha p^2 \beta \{r_2^2 \phi(2r_1) - 2r_1 r_2 \phi(r_1+r_2) + r_1^2 \phi(2r_2)\} + \beta^2 p \{r_2 \phi(r_1) - r_1 \phi(r_2)\} + \alpha \beta p \{r_2 \phi(r_1) - r_1 \phi(r_2)\} - \beta^2$$

$$E(a_{12}b_1) = \beta p^2 \{r_2 \phi(2r_1) - (r_1+r_2) \phi(r_1+r_2) + r_1 \phi(2r_2)\} - p \beta \{\phi(r_1) - \phi(r_2)\}$$

$$E(a_{21}b_1) = -\alpha \beta r_1 r_2 p^2 \{r_2 \phi(2r_1) - (r_1+r_2) \phi(r_1+r_2) + r_1 \phi(2r_2)\} + \alpha \beta p r_1 r_2 \{\phi(r_1) - \phi(r_2)\}$$

$$E(a_{22}b_1) = \beta p^2 \{r_1 r_2 \phi(2r_1) - (r_1^2+r_2^2) \phi(r_1+r_2) + r_1 r_2 \phi(2r_2)\} - p \beta \{r_1 \phi(r_1) - r_2 \phi(r_2)\}$$

$$E(a_{11}b_2) = -\alpha \beta p^2 r_1 r_2 \{r_2 \phi(2r_1) - (r_1+r_2) \phi(r_1+r_2) + r_1 \phi(2r_2)\} + \beta^2 p r_1 r_2 \{\phi(r_1) - \phi(r_2)\}$$

$$E(a_{12}b_2) = \beta p^2 r_1 r_2 \{\phi(2r_1) - 2\phi(r_1+r_2) + \phi(2r_2)\}$$

$$E(a_{21}b_2) = -\alpha \beta (r_1 r_2 p)^2 \{\phi(2r_1) - 2\phi(r_1+r_2) + \phi(2r_2)\}$$

$$E(a_{22}b_2) = \beta p^2 r_1 r_2 \{r_1 \phi(2r_1) - (r_1+r_2) \phi(r_1+r_2) + r_2 \phi(2r_2)\}$$

We must now solve  $E(A'WA) - W = -Q$  for  $W$ , where

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{Let } W = \begin{pmatrix} W_1 & W_2 & W_4 \\ W_2 & W_3 & W_5 \\ W_4 & W_5 & W_6 \end{pmatrix} \quad \hat{W} = \begin{pmatrix} W_1 & W_2 \\ W_2 & W_3 \end{pmatrix} \quad \underline{W} = \begin{pmatrix} W_4 \\ W_5 \end{pmatrix}$$

It can be shown that

$$E(A'WA) = \left( \frac{E(A'\hat{W}A)}{E(b'\hat{W}A) + E(C\underline{W}'A)} \mid \frac{E(A'\hat{W}b) + E(A'\underline{W}C)}{E(b'\hat{W}b) + 2E(Cb')\underline{W} + E(C^2)W_6} \right)$$

Thus, the solution of  $E(A'WA) - W = -Q$  may be carried out via the consecutive solution of

$$E(A'\hat{W}A) - \hat{W} = - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{for } \hat{W}$$

$$E(b'\hat{W}A) + E(C\underline{W}'A) - \underline{W} = \begin{pmatrix} 0 & 0 \end{pmatrix} \quad \text{for } \underline{W}$$

$$E(b'\hat{W}b) + 2E(Cb')\underline{W} + E(C^2)W_6 - W_6 = 0 \quad \text{for } W_6.$$

This is a simple matter for the digital computer. From theorems previously developed

$$\lim_n E \begin{pmatrix} x_n \\ y_n \\ u_n \end{pmatrix}' Q \begin{pmatrix} x_n \\ y_n \\ u_n \end{pmatrix} = E \begin{pmatrix} 0 \\ 0 \\ h_n \end{pmatrix}' W \begin{pmatrix} 0 \\ 0 \\ h_n \end{pmatrix}$$

$$\text{or } \lim_n E(x_n' x_n) = E(h_n^2) W_6 = [1 - \phi(2r_3)] W_6.$$



As an illustration, consider the sampling device to be subjected to uniform jitter,  $D$ , about nominal sampling time,  $T$ . Let

$$K = 10.0$$

$$T = 1.0$$

$$D = .2$$

$$r_1 = -1.$$

$$r_2 = -2.$$

$$r_3 = -.5$$

With the above data and using a straightforward Fortran program, we find that

$$\lim_{n \rightarrow \infty} E(x_n^2) = 1.519 \times 10^2 = 151.9.$$

At first glance, this might seem unusually large since the variance of  $u$  is  $\sqrt{(2)(.5)} = 1$ . However, note that the gain  $K$  is in the forward loop and so with  $K = 10$  the input variance is 100. With  $K$  in the feedback loop, the asymptotic value of the output's second moment was computed to be 1.519.

#### RANDOMLY SAMPLED LURE PROBLEM

The methods developed for linear systems will now be utilized to analyze a system with a scalar non-linear feedback element. For a discussion of the deterministic problem, see Lefshetz (ref. 12).

The system equations are (see Figure 12):

$$\dot{x} = Ax - mf(\sigma_n) \quad t \in [t_n, t_{n+1}) \quad (23)$$

$$\dot{\sigma} = c'x \quad (24)$$

where  $x$ ,  $m$ , and  $c$  are  $N$  vectors and  $\sigma$  is a scalar; where  $A$  is an asymptotically stable  $N \times N$  matrix;  $\sigma_n = \sigma(t_n)$ , and the following assumptions are placed on the scalar function  $f$ :

$$f(0) = 0$$

$$\sigma f(\sigma) > 0 \quad \sigma \neq 0$$

$$\left| \frac{df}{d\sigma} \right| \leq \mu$$

$$I(\sigma) = \int_0^{\sigma} f(\hat{\sigma}) d\hat{\sigma} \geq 0$$

$$I(\sigma) \rightarrow \infty \text{ as } \sigma \rightarrow \pm\infty$$

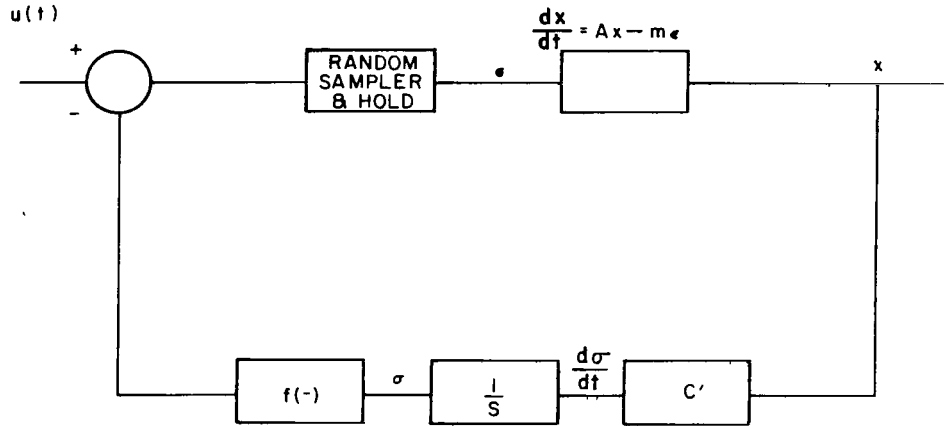


Figure 12.- The randomly-sampled Lure problem

A straightforward integration of Eqs. (23) and (24) yields

$$x_{n+1} = A_n x_n + a_n f(\sigma_n)$$

$$\sigma_{n+1} = \sigma_n - b_n x_n + r_n$$

where

$$A_n = e^{A\Delta_n}$$

$$a_n = A^{-1}(I - e^{A\Delta_n})m$$

$$b_n' = c'A^{-1}(I - e^{A\Delta_n})$$

$$r_n = c'A^{-1}\left[\Delta_n I + A^{-1}(I - e^{A\Delta_n})\right]m$$

We consider the Lyapunov function

$$V(x, \sigma) = x'Hx + q \int_0^{\sigma} f(\alpha) d\alpha$$

where  $q > 0$  and  $H$  is positive definite. This has been used by Szego and Pearson (ref. 13) for the discrete time deterministic Lure problem. The following applies to the system of Figure 11. Suppose that there is an  $H > 0$  and a  $q > 0$  such that  $C_1 < 0$  and

$$\rho > dC_1^{-1}d'$$

where

$$C_1 = \left[ E A_n' H A_n - H + \frac{\mu}{2} q b_n b_n' \right]$$

$$d' = E \left[ a_n' H A_n - \frac{\mu}{2} q r_n b_n' - \frac{q}{2} b_n' \right]$$

$$\rho = E \left[ a_n' H a_n + q r_n + \frac{\mu q}{2} r_n^2 \right]$$

Then the system is asymptotically stable wpl.

One can obtain bounds for intersample behavior and can obtain estimates of asymptotic behavior when inputs are present, but these questions will not be considered here.

## CONCLUSION

Systems which can be modelled as randomly sampled linear systems have been studied by a stochastic Lyapunov function method. Stability criteria have been presented when no input is present and asymptotic behavior of driven systems has been studied. The conditions obtained are straightforward to apply, as the discussion of the computer implementation has indicated. In the present form, the conditons are directly applicable to the study of systems with jitter or skipping to determine the effect of these imperfections. However, prior to use in other practical situations the following modifications should be incorporated. If one wants to analyze remote control systems, time delays in the control loop must be considered; also, for application to compensatory tracking functions one should give serious consideration to dead-zone non-linearities in the forward path. Both of the above constitute areas of future research.

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## APPENDIX

### SOME RELEVANT DEFINITIONS AND THEOREMS

A few definitions and theorems will be present which are used throughout this paper.

Definition A.1 - Mean-Square Stability of a Random Process.-  $x(t)$ .  $x(t)$  is mean-square stable if  $\forall \epsilon > 0$ ,  $\exists \delta(\epsilon) > 0$  such that if  $||x(t_0)|| < \delta$ , then  $E(||x(t)||^2) < \epsilon$ , for all  $t > t_0$ .

Definition A.2 - Asymptotic Mean-Square Stability.-  $x(t)$  is asymptotically mean-square stable if

(a)  $x(t)$  is mean square stable

$$(b) \lim_{t \rightarrow \infty} E[||x(t)||^2] = 0$$

Definition A.3 - Stability With Probability One.-  $x(t)$  is stable with probability one if  $\forall \epsilon > 0$ ,  $\epsilon' > 0$ ,  $\exists \delta(\epsilon, \epsilon') > 0$  such that if  $||x(t_0)|| > \delta$ , then

$$P[\sup_{t \geq t_0} ||x(t)|| > \epsilon'] < \epsilon$$

Definition A.4 - Asymptotic Stability With Probability One.-  $x(t)$  is a asymptotically stable wpl if

(a)  $x(t)$  is stable wpl

$$(b) \lim_{t \rightarrow \infty} x(t) = 0 \text{ wpl}$$

Definition A.5 - Non-Negative Supermartingale.- Let  $x_n$  be a discrete Markov process and let  $V_n = V(x_n) \geq 0$  have the property that

$$E[V_{n+1} | x_n] - V_n = -K(x_n) \leq 0.$$

Then the sequence  $\{V_n\}$  is called a non-negative supermartingale sequence.

Lemma A.1 - (Doob Martingale Convergence Theorem).- Suppose that  $\{V_n\}$  is a non-negative supermartingale sequence. Then there is a  $\bar{V} \geq 0$  such that  $V_n \rightarrow \bar{V}$  wpl, and

$$P\{\sup_{n \in [0, \infty]} V_n \geq \epsilon \mid x_0 = x\} \leq \frac{1}{\epsilon} V(x)$$

Also  $\left[ E \sum_0^{\infty} K_n \mid x_0 = x \right] \leq V_0$  and  $K_n \rightarrow 0$  wpl.

(For proof, see references A-1 and A-2.)

Lemma A.2 - Let  $\{x_n\}$  be a Markov process and  $V(x)$  a non-negative function. Suppose  $E[V_{n+1} \mid x_n] - V_n \leq -\delta V_n$ ,  $\delta > 0$ .

Then  $P\left\{ \sup_{n \in [0, \infty]} V_n (1-\delta)^{-n} \geq \varepsilon \mid x_0 = x \right\} \leq \frac{1}{\varepsilon} V(x_0)$

and  $\lim_n E[V_n (1-\delta_1)^n \mid x_0 = x] = 0$  where  $\delta_1 \in (0, \delta)$ .

Now, we shall restrict ourselves to first-order systems and make two more definitions for that case.

Definition A.6 - Stability of Order  $s$  ( $s > 0$ ).-  $x(t)$  is stable of order  $s$  if  $\forall \varepsilon > 0$ ,  $\exists \delta(\varepsilon) > 0$  such that if  $|x_0|^s < \delta$ , then  $E(|x(t)|^s) < \varepsilon$ .

Definition A.7 - Asymptotic Stability of Order  $s$  ( $s > 0$ ).-  $x(t)$  is asymptotically stable of order  $s$  if

(1)  $x(t)$  is stable of order  $s$

(2)  $\lim_{t \rightarrow \infty} \left[ E |x(t)|^s \right] = 0$

#### REFERENCES

- A-1 Doob, J. L.: Stochastic Processes. J. Wiley, 1953.
- A-2 Kushner, H. J.: On the Stability of Stochastic Dynamical Systems. Proceedings, National Academy of Science, vol. 53, 1965, p. 8.



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